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OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Monday 18 October 2021 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours
plus your additional time allowance

DC (CJ/SG) 301260/1

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INSTRUCTIONS

Do NOT send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

This Insert contains the article for Section B.

ADDING ARCTANGENTS

Where does the name 'arctangent' come from?

The two commonly used ways to denote the angle which has a tangent x are $\tan^{-1}x$ and $\arctan x$. The first of these is related to inverse function notation, $\mathrm{f}^{-1}(x)$. Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle; $\arctan x$ is the arc whose tangent is x.

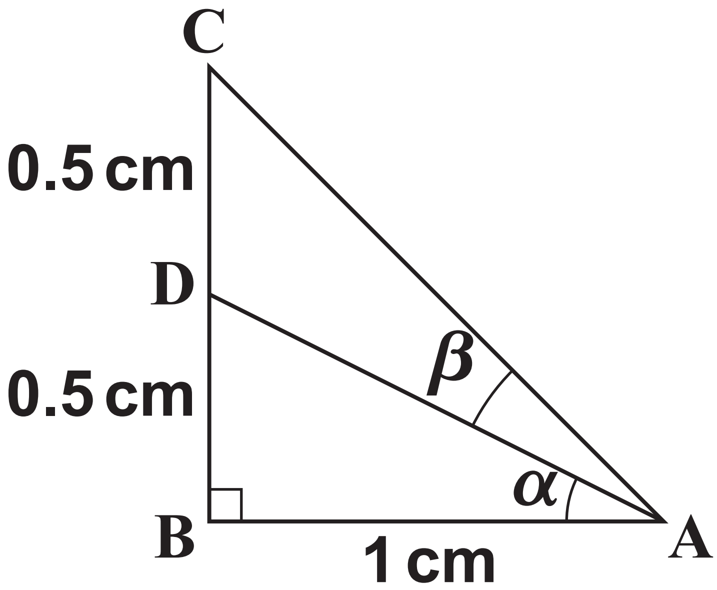
An interesting result

It can be shown that

$$\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1.$$

Consider the diagram in FIG. C1.

FIG. C1



Triangle ABC is right-angled at B. 15
AB = BC = 1 cm.
D is the midpoint of BC.

Using triangle ABD, $\tan\alpha = \dfrac{\mathrm{DB}}{\mathrm{BA}} = \dfrac{1}{2}$ so $\alpha = \arctan\left(\dfrac{1}{2}\right)$.

Using triangle ABC, $\tan(\alpha+\beta) = 1$ so $\alpha+\beta = \arctan 1$.

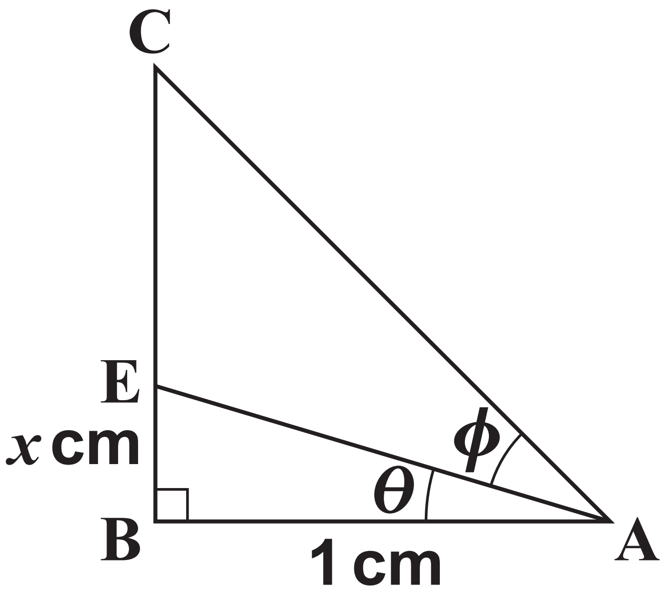
Hence $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = 1$.

Using $\tan\alpha = \dfrac{1}{2}$ and finding $\tan\beta$, it follows that $\beta = \arctan\left(\dfrac{1}{3}\right)$,

which gives the required result that $\arctan\left(\dfrac{1}{2}\right) + \arctan\left(\dfrac{1}{3}\right) = \arctan 1$.

Generalising the result

FIG. C2



Triangle ABC in FIG. C2 is the same as triangle ABC in FIG. C1 but E is a point on BC such that $\mathrm{EB} = x$ cm and $\theta = \arctan x$.

Following the same method as above,

$$\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \arctan 1.$$

The arctan addition formula

The arctangent addition formula is a further generalization:

$\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$, as long as $xy < 1$.

This result is equivalent to the addition formula $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ where $\alpha = \arctan x$ and $\beta = \arctan y$.

To see why the restriction $xy < 1$ is necessary, consider what happens if $xy \geqslant 1$.

Clearly, $\dfrac{x+y}{1-xy}$ is undefined when $xy = 1$, so the formula does not apply in this case.

Suppose next that $xy > 1$, and that x and y are both positive; in this case $y > \frac{1}{x}$.

For any positive x,
$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$.
$y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$ so it follows that $\arctan x + \arctan y > \frac{\pi}{2}$.

However, $\arctan\left(\frac{x+y}{1-xy}\right)$ cannot be greater than $\frac{\pi}{2}$ as the range of the arctan function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The formula
$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$
therefore cannot be valid in this case.

A similar argument can be used to show that the formula cannot be valid when $xy > 1$ and x and y are both negative.

If $xy > 1$, the arctangent addition formula needs to be adapted, as shown below.

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi,$$

when $xy > 1$ and $x, y < 0$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi,$$

when $xy > 1$ and $x, y > 0$

Some additional results

- For n a positive integer, $\arctan\left(\frac{1}{n+1}\right)+\arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right)$; this follows directly from the arctan addition formula in lines 37–38.

- $\arctan 1+\arctan 2+\arctan 3=\pi$. This can be proved by using $\arctan x+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$ together with $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

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